Linear Programming Models: Graphical and Computer Methods

Introduction

Linear programming (LP) is

a widely used mathematical modeling technique designed to help managers in planning and decision making relative to resource allocation.

LP is a technique that helps in resource allocation decisions.

Programming refers to

- modeling and solving a problem mathematically.

Examples of Successful LP Applications

1. Development of a production schedule that will satisfy future demands for a firm's production while *minimizing* total production and inventory costs
2. Selection of product mix in a factory to make best use of machine-hours and labor-hours available while *maximizing* the firm's products

Examples of Successful LP Applications *(continued)*

3. Determination of grades of petroleum products to yield the *maximum* profit
4. Selection of different blends of raw materials to feed mills to produce finished feed combinations at *minimum* cost
5. Determination of a distribution system that will *minimize* total shipping cost from several warehouses to various market locations

Linear Programming Problems

All LP problems have 4 properties in common:

- All problems seek to maximize or minimize some quantity (the objective function).
- The presence of restrictions or constraints limits the degree to which we can pursue our objective.
- There must be alternative courses of action to choose from.
- The objective and constraints in linear programming problems must be expressed in terms of *linear* equations or inequalities.

5 Basic Assumptions of Linear Programming

1. **Certainty:**
   - numbers in the objective and constraints are known with *certainty* and do not change during the period being studied
2. **Proportionality:**
   - exists in the objective and constraints
3. **Additivity:**
   - the total of all activities equals the sum of the individual activities

Basic Assumptions of Linear Programming *(continued)*

4. **Divisibility:**
   - solutions need not be in whole numbers (integers)
   - solutions are divisible, and may take any fractional value
5. **Non-negativity:**
   - all answers or variables are greater than or equal to (≥) zero
   - negative values of physical quantities are impossible

Formulating Linear Programming Problems

- Formulating a linear program involves developing a mathematical model to represent the managerial problem.
- Once the managerial problem is understood, begin to develop the mathematical statement of the problem.
- The steps in formulating a linear program follow on the next slide.

Steps in LP Formulations

1. Completely understand the managerial problem being faced.
2. Identify the ***objective*** and the constraints.
3. Define the ***decision variables***.
4. Use the decision variables to write mathematical expressions for the objective function and the constraints.

Formulating Linear Programming Problems *(continued)*

The *Product Mix Problem*

- Two or more products are usually produced using limited resources such as
  - personnel, machines, raw materials, and so on.
- The profit that the firm seeks to maximize is based on the profit contribution per unit of each product.
- The company would like to determine how many units of each product it should produce so as to maximize overall profit given its limited resources.
- A problem of this type is formulated in the following example on the next slide.

Furniture Company Data

Hours Required to Produce One Unit

Department	*T* Tables	*C* Chairs	Available Hours This Week
• Carpentry	4	3	240
• Painting &Varnishing	2	1	100

Mathematical formulation:

Profit Amount $7 $5

Constraints: $4T + 3C \leq 240$ (Carpentry)

$2T + 1C \leq 100$ (Paint & Varnishing)

$T \geq 0$ (1st nonnegative cons)

$C \geq 0$ (2nd nonnegative cons)

Max. Objective, z: $7T + 5C$

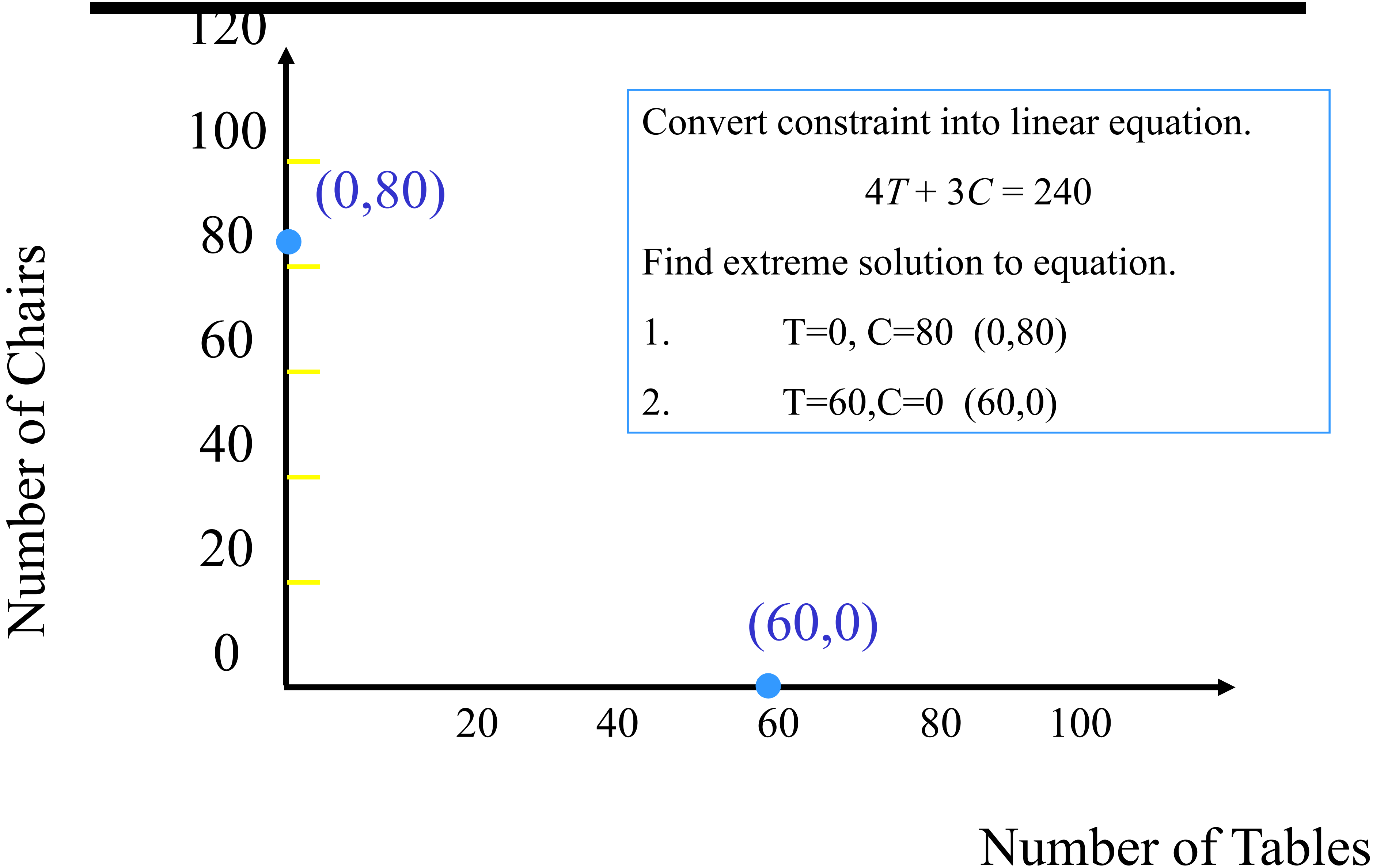
Graphical Solution Approach

The easiest way to solve a small LP problem, is with the *graphical solution approach.*

The graphical method works only when there are two decision variables, but it provides valuable insight into how larger problems are structured.

When there are more than two variables, it is not possible to plot the solution on a two-dimensional graph; a more complex approach is needed.

Showing Constraint on Graph
Carpentry Constraint (4T+3C <=240)



Convert constraint into linear equation.

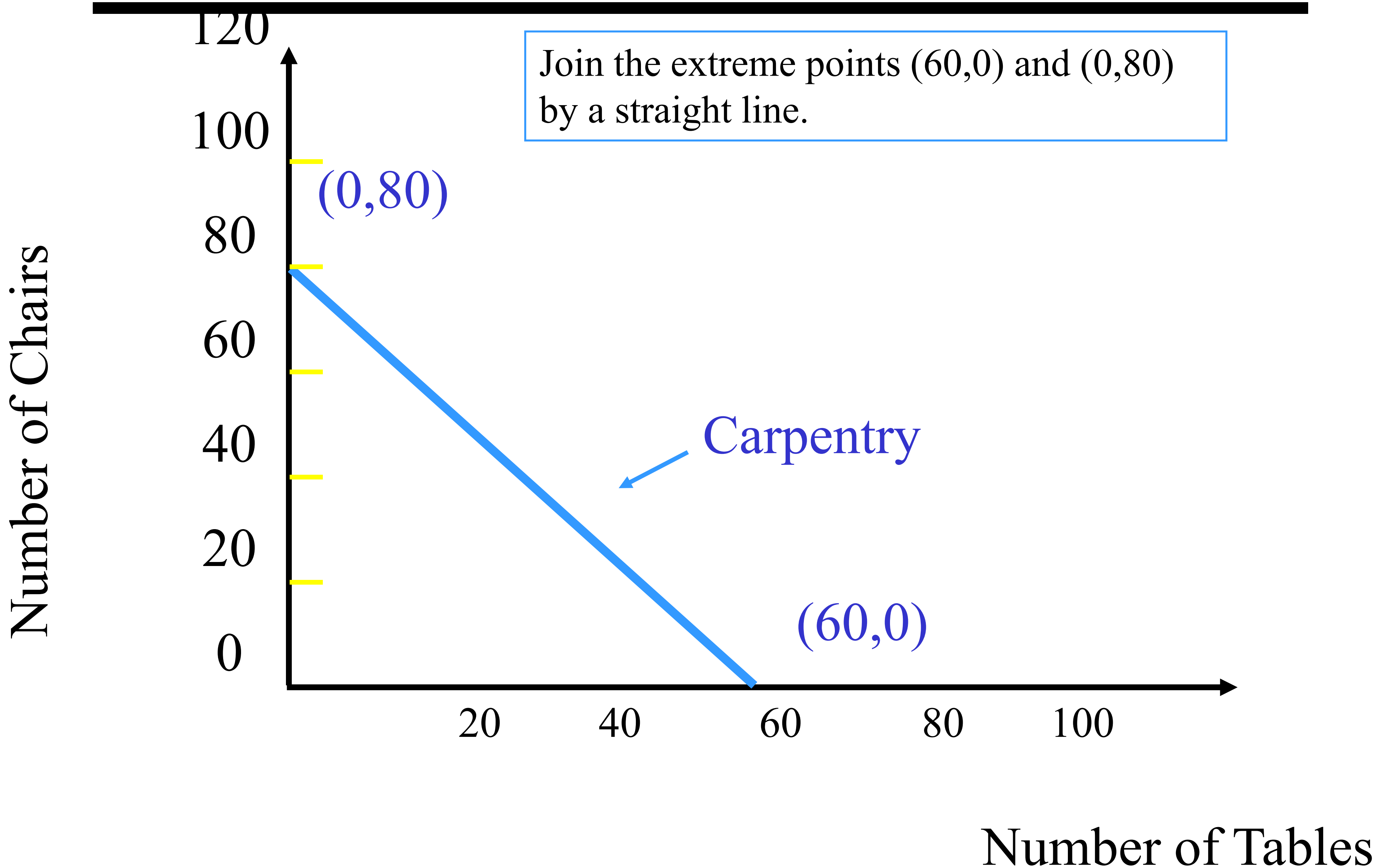
$$4T + 3C = 240$$

Find extreme solution to equation.

1. T=0, C=80 (0,80)
2. T=60,C=0 (60,0)

Carpentry Constraint (4T+3C <=240)

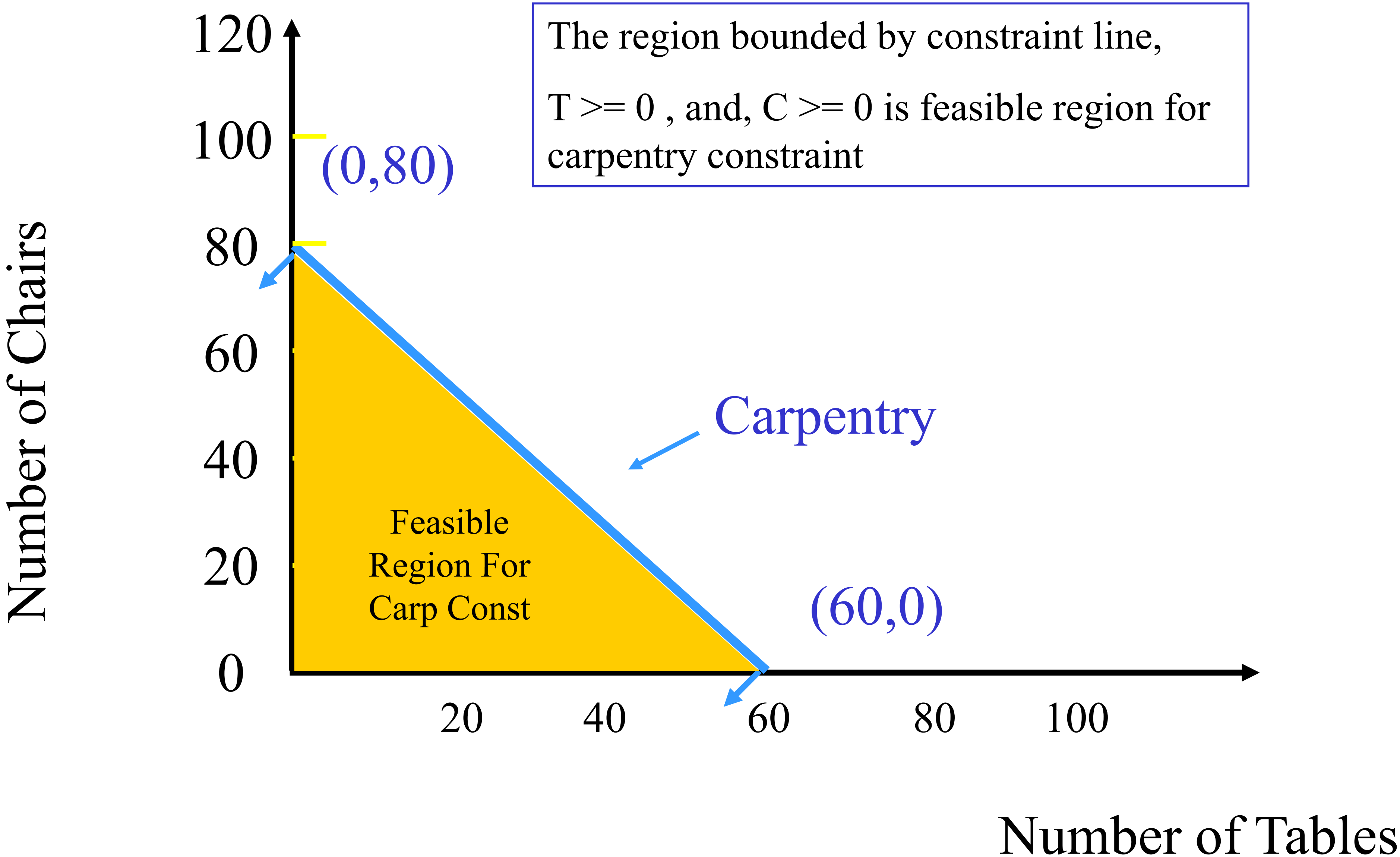
Join the extreme points (60,0) and (0,80) by a straight line.



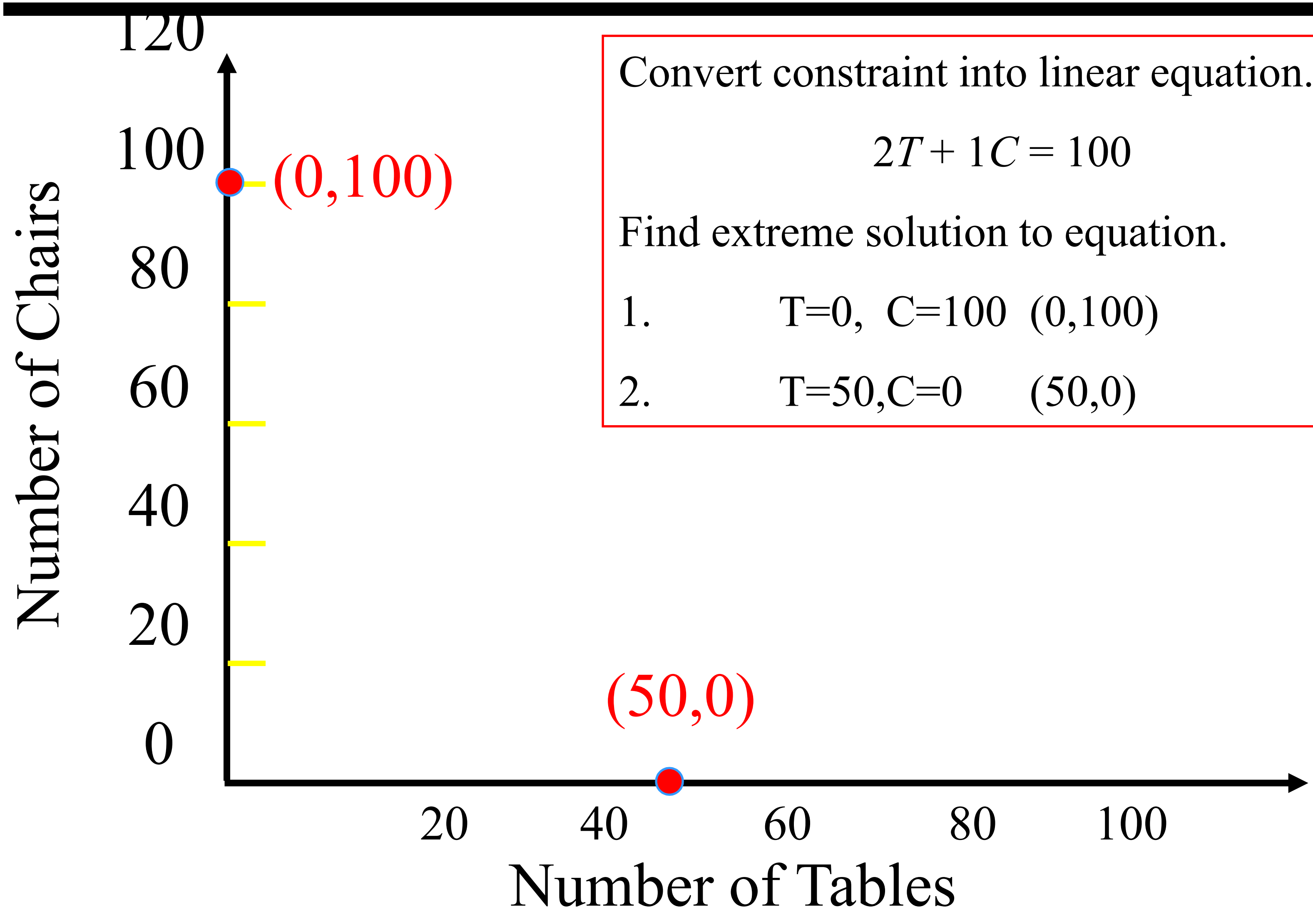
Carpentry Constraint (4T+3C <=240)

The region bounded by constraint line,

T >= 0 , and, C >= 0 is feasible region for carpentry constraint



Painting/Varnishing Constraint (2T+ 1 C <= 100)



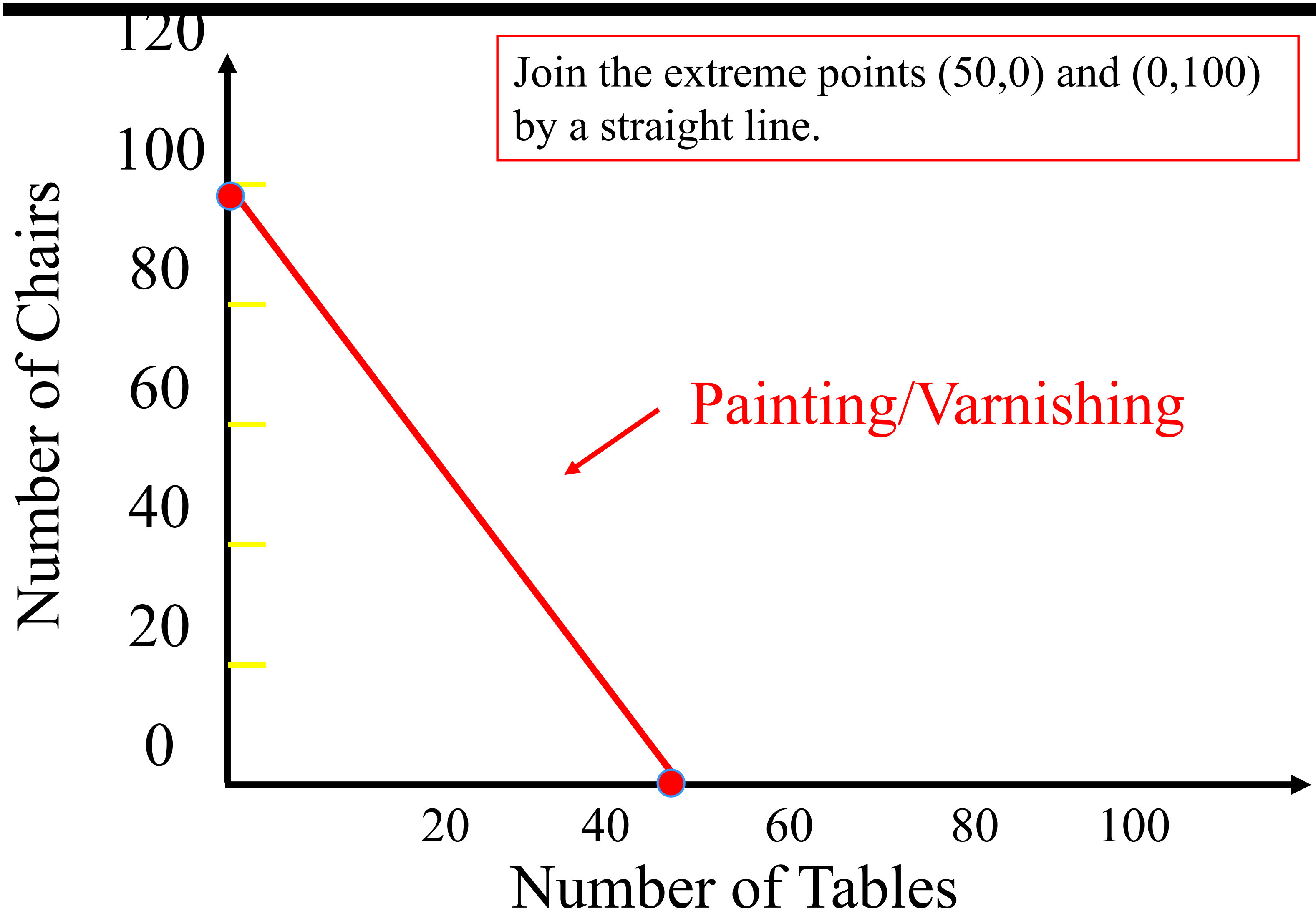
Convert constraint into linear equation.

$$2T + 1C = 100$$

Find extreme solution to equation.

1. T=0, C=100 (0,100)
2. T=50,C=0 (50,0)

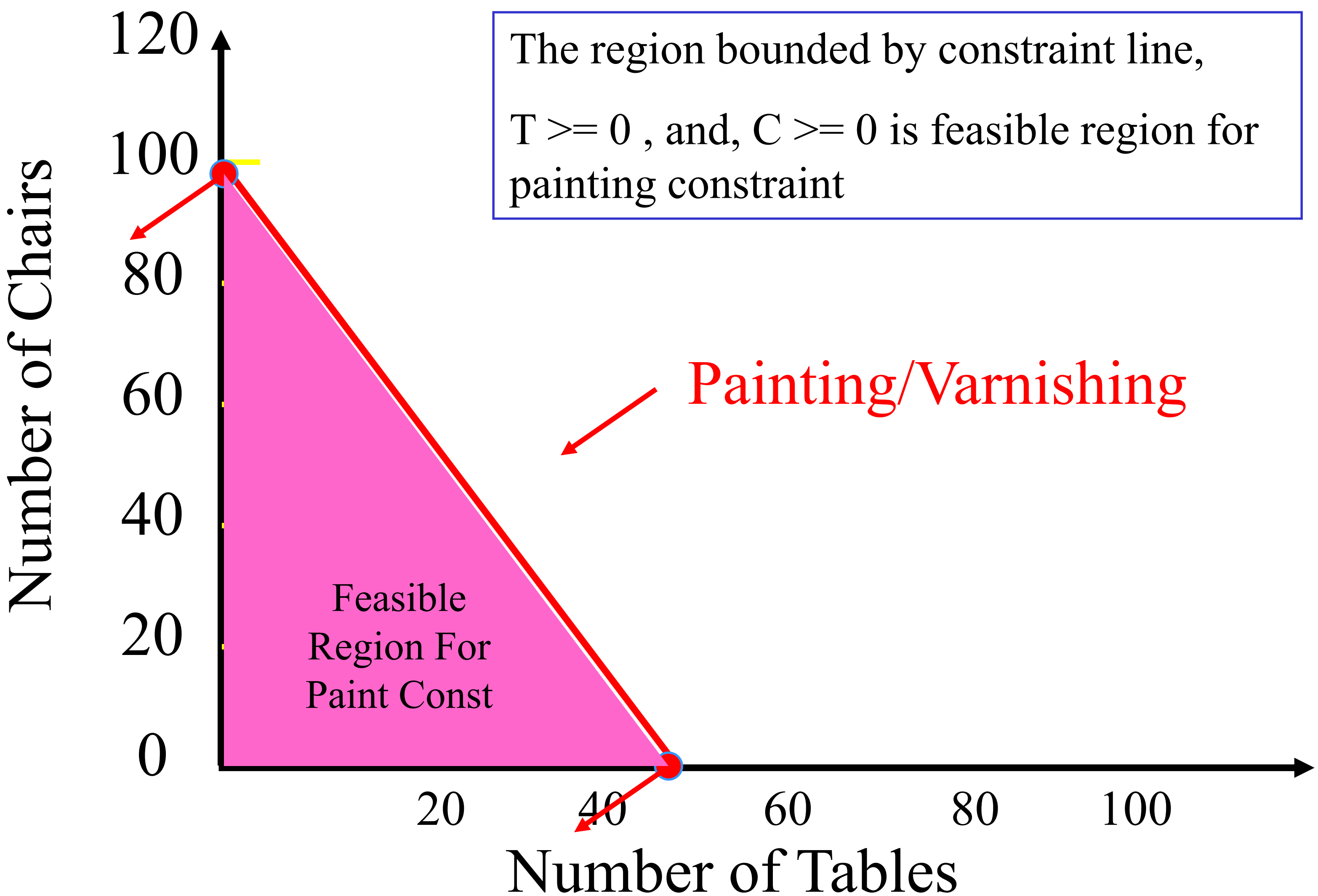
Painting/Varnishing Constraint (2T+ 1 C <= 100)



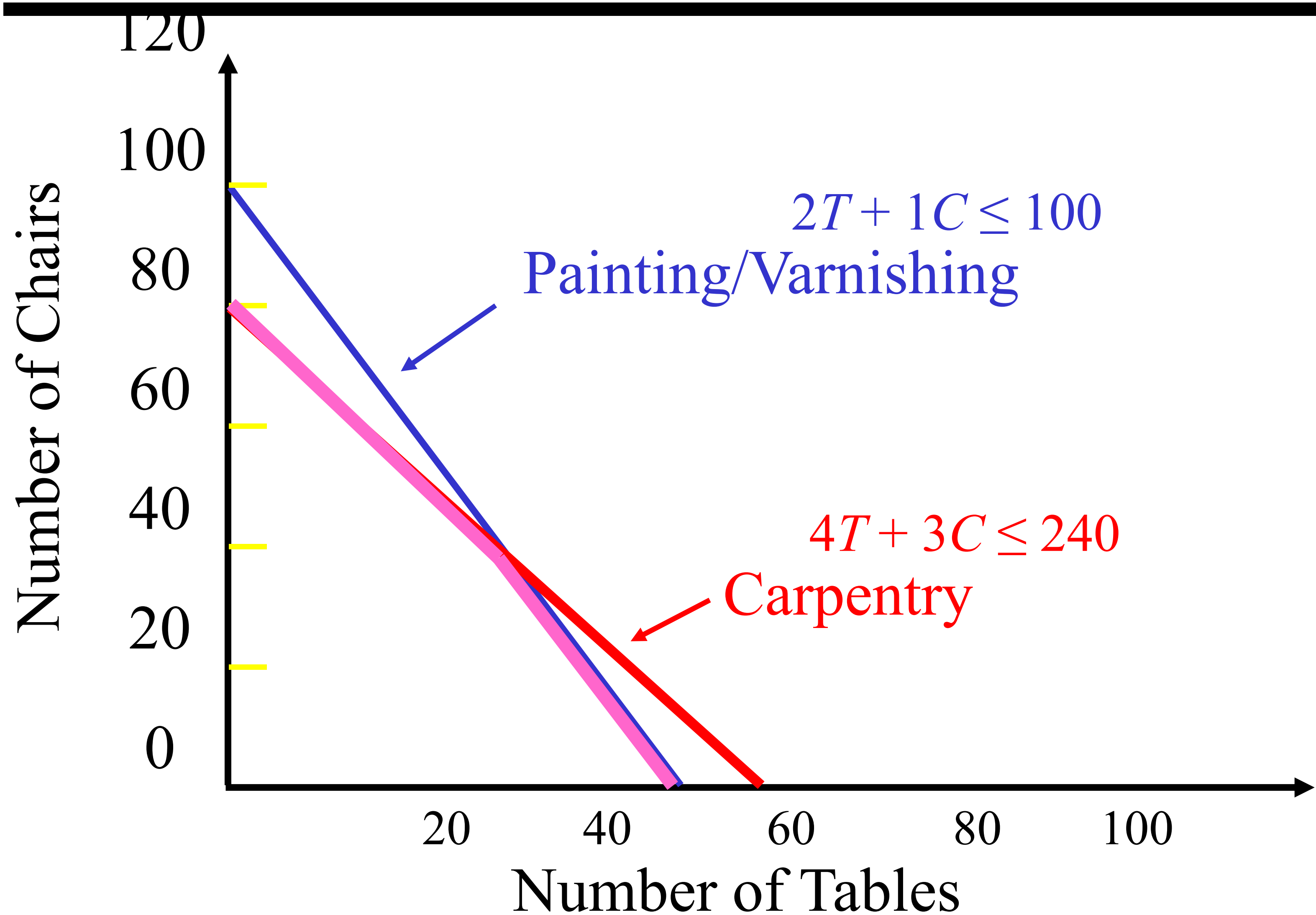
Painting/Varnishing Constraint (2T+ 1 C <= 100)

The region bounded by constraint line,

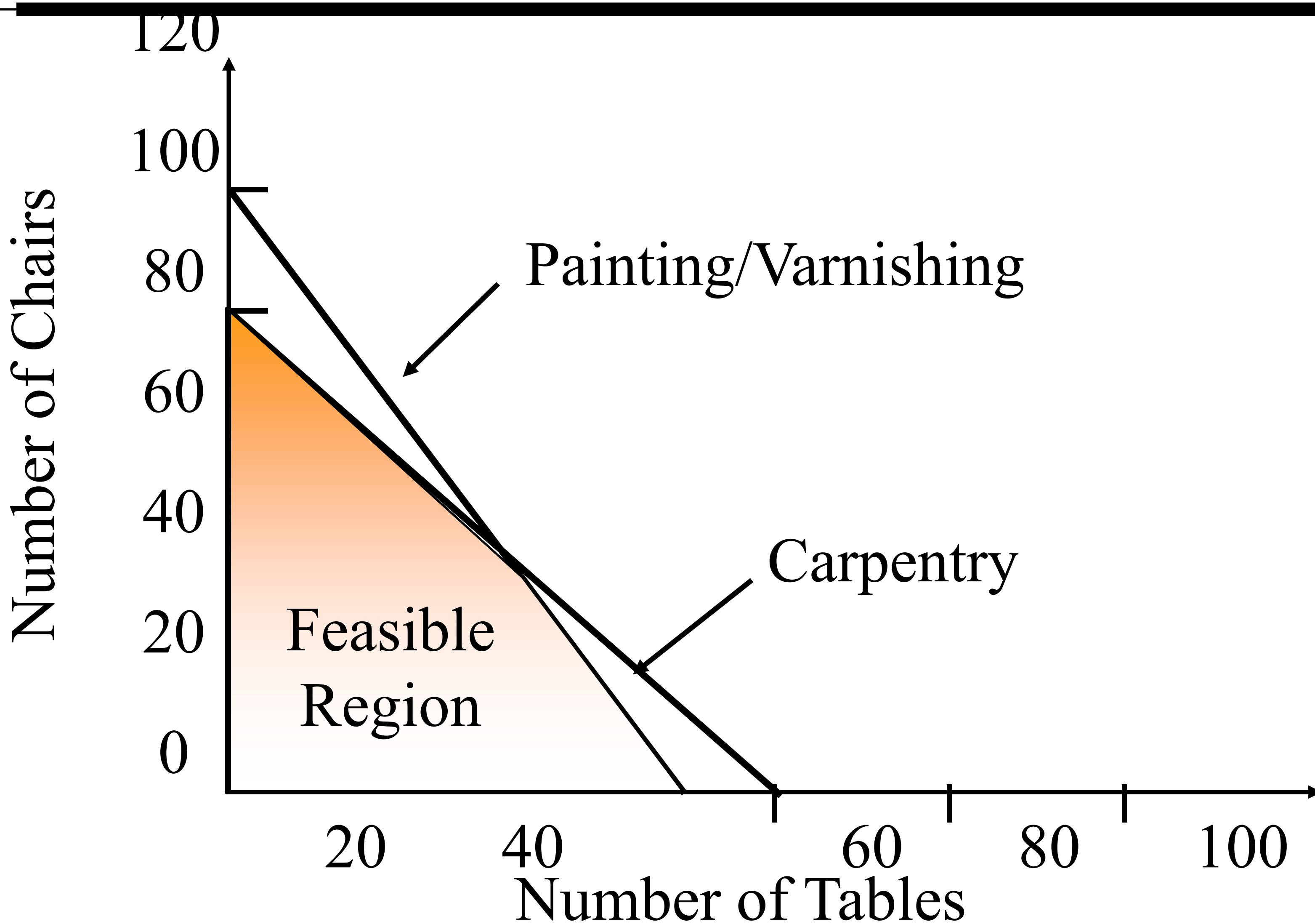
T >= 0 , and, C >= 0 is feasible region for painting constraint



Superimposing Two Constraints



Feasible Region



Finding Optimal Solution

Corner Point Solution Method

- It involves looking at the profit at every corner point of the feasible region
- The mathematical theory behind LP is that the optimal solution must lie at one of the corner points in the feasible region

Corner Point

Corner Point Solution Method, Summary

1. Graph all constraints and find the feasible region.
2. Find the corner points of the feasible region.
3. Compute the profit (or cost) at each of the feasible corner points.
4. Select the corner point with the best value of the objective function found in step 3. This is the optimal solution.

Furniture Company Corner Point

Corner Point Solution Method

- The feasible region for the Flair Furniture Company problem is a four-sided polygon with four corner, or extreme, points.
- These points are labeled 1 ,2 ,3 , and 4 on the next graph.
- To find the (T, C) values producing the maximum profit, find the coordinates of each corner point and test their profit levels.

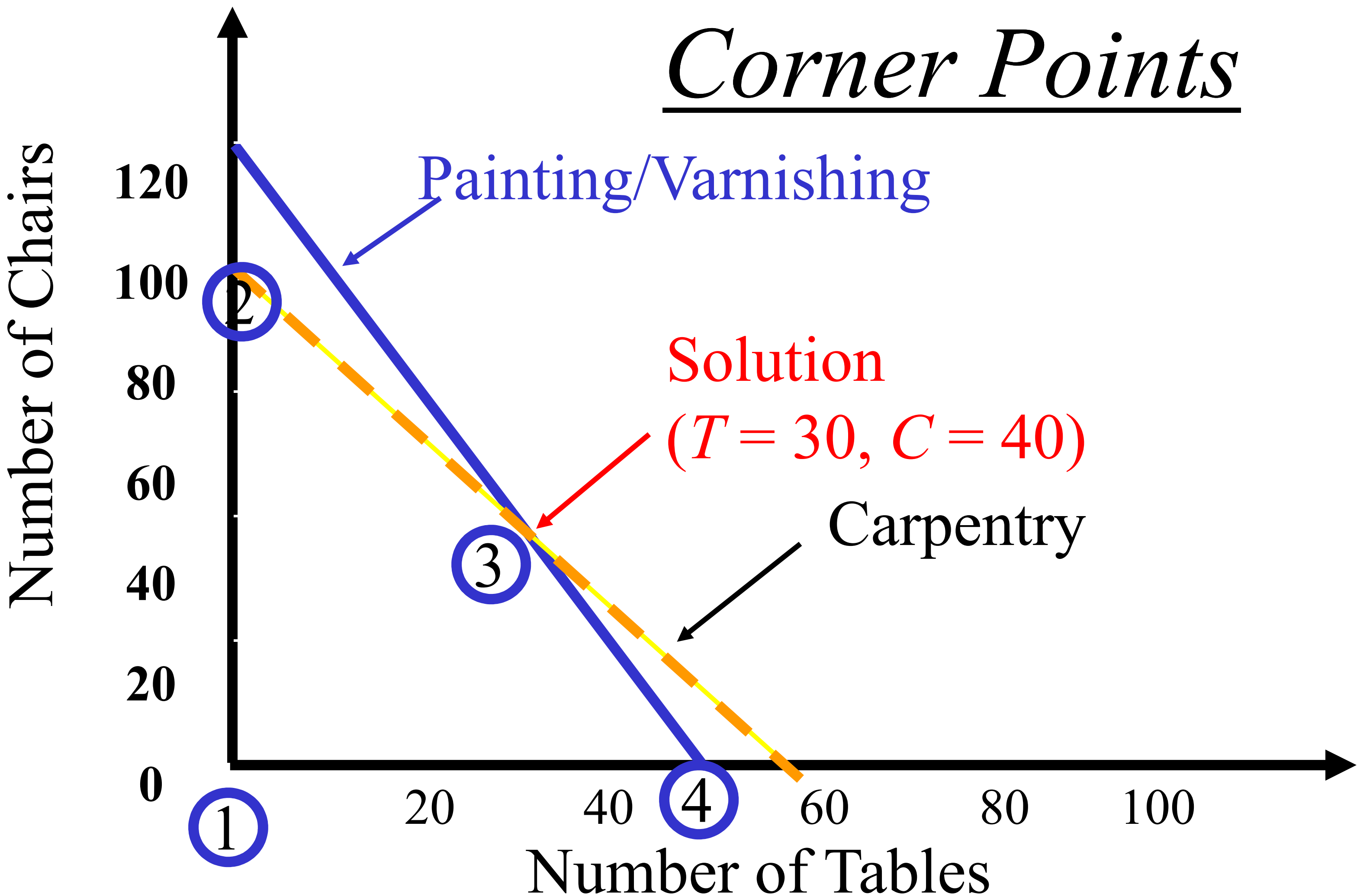
Point 1: $(T = 0, C = 0)$ profit = \$7(0) + \$5(0) = \$0

Point 2: $(T = 0, C = 80)$ profit = \$7(0) + \$5(80) = \$400

Point 3: $(T = 30, C = 40)$ profit = \$7(30) + \$5(40) = \$410

Point 4 : $(T = 50, C = 0)$ profit = \$7(50) + \$5(0) = \$350

Furniture Company Optimal Solution



FINDING OPTIMAL SOLUTION

Isoprofit Line Solution Method

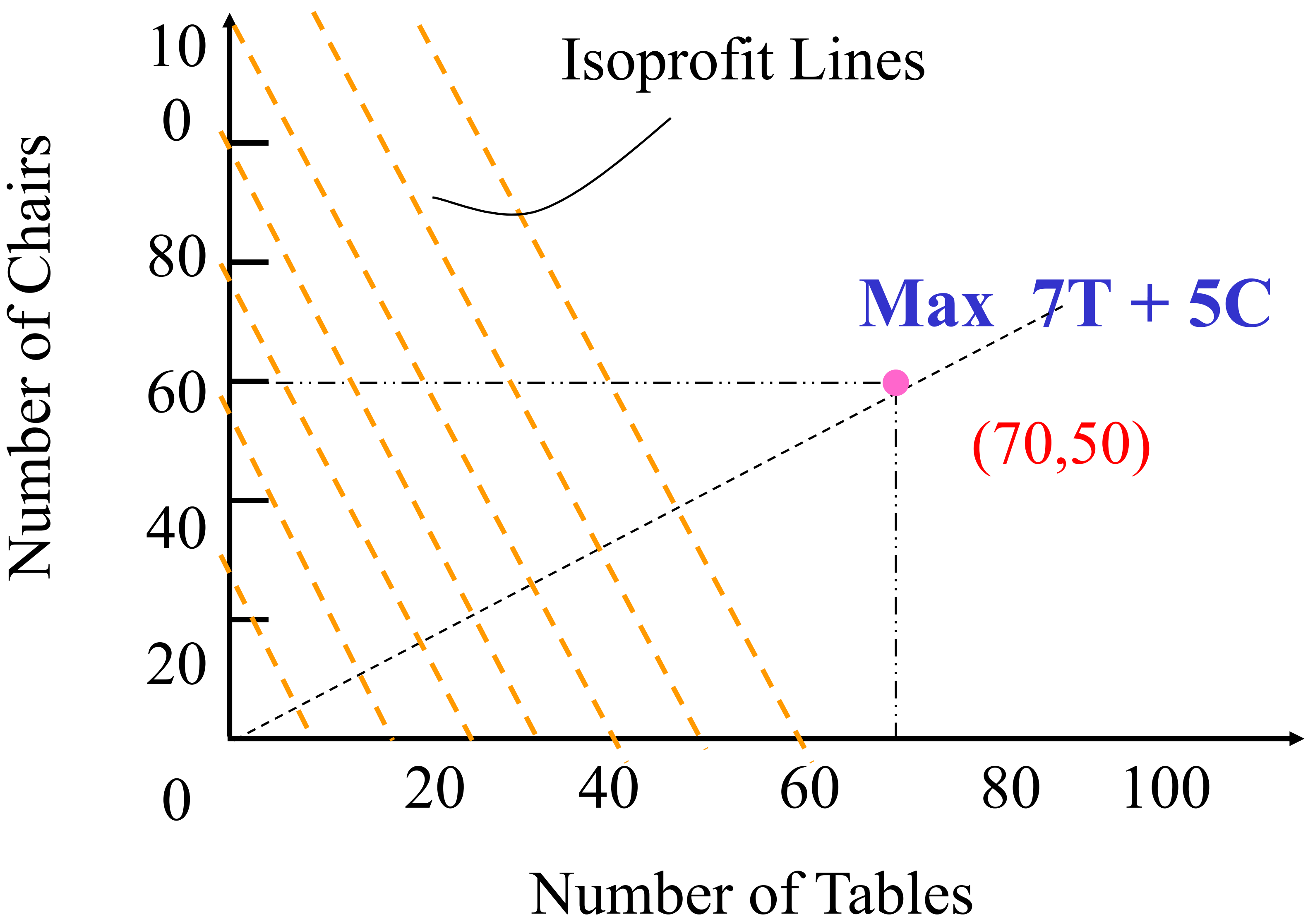
- The objective function is just the equation of a line called an *Isoprofit line.*
  - Draw a straight line that would yield a total profit of $210.
- To plot the profit line, proceed exactly as done to plot a constraint line:
  - First, let *T* = 0 and solve for the point at which the line crosses the *C* axis.
  - Then, let *C* = 0 and solve for *T*.
  - ✓ $210 = $7(0) + $5(C) → C = 42 chairs
- Then, let C = 0 and solve for T.
  - ✓ $210 = $7(T) + $5(0) → T = 30 tables

Furniture Company Isoprofit Lines

Isoprofit Line Solution Method

- Next connect these two points with a straight line. This profit line is illustrated in the next slide.
- All points on the line represent feasible solutions that produce an approximate profit of \$210
- Obviously, the isoprofit line for \$210 does not produce the highest possible profit to the firm.
- Try graphing more lines, each yielding a higher profit.
- Another equation, $\$420 = \$7T + \$5C$, is plotted in the same fashion as the lower line.

Furniture Company Isoprofit Lines

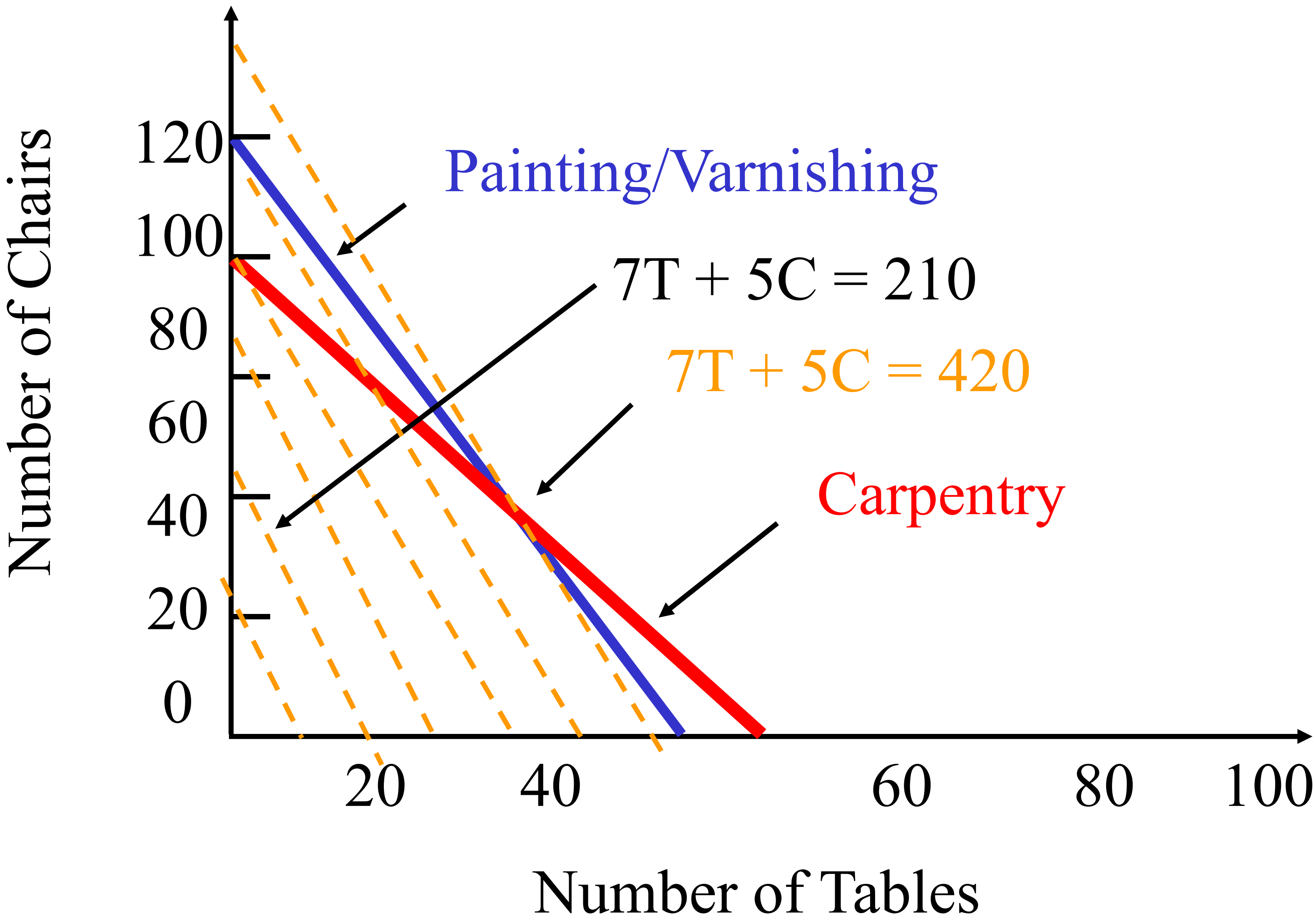


Furniture Company Isoprofit Lines

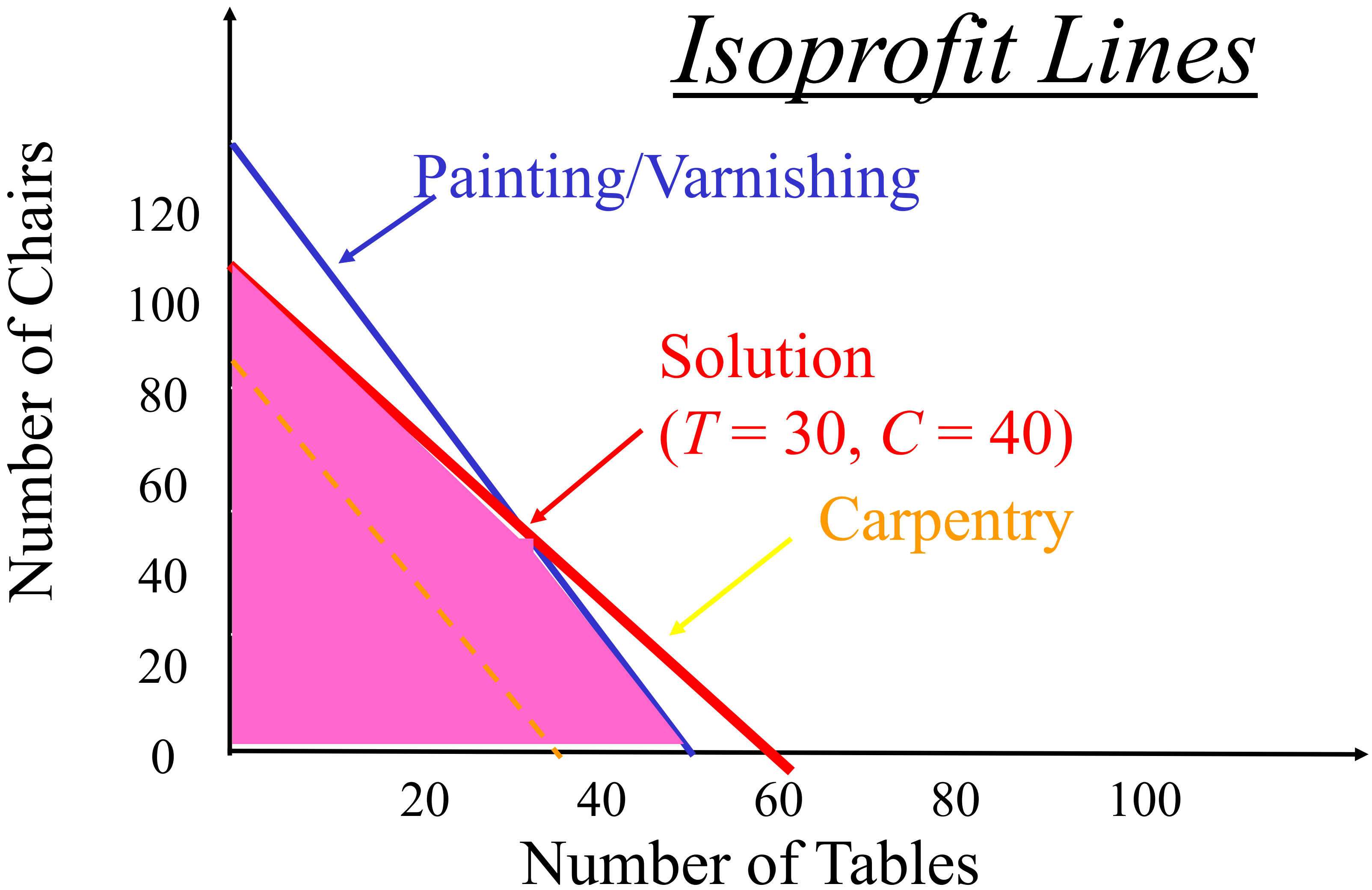
Isoprofit Line Solution Method

- When $T = 0$,
  - ✓ $420 = $7(0) + 5(C) → C = 84 chairs
- When $C = 0$,
  - ✓ $420 = $7(T) + 5(0) → T = 60 tables
- This line is too high to be considered as it no longer touches the feasible region.
- The highest possible isoprofit line is illustrated in the second following slide. It touches the tip of the feasible region at the corner point ($T = 30$, $C = 40$) and yields a profit of $410.

Furniture Company Isoprofit Lines



Furniture Company Optimal Solution

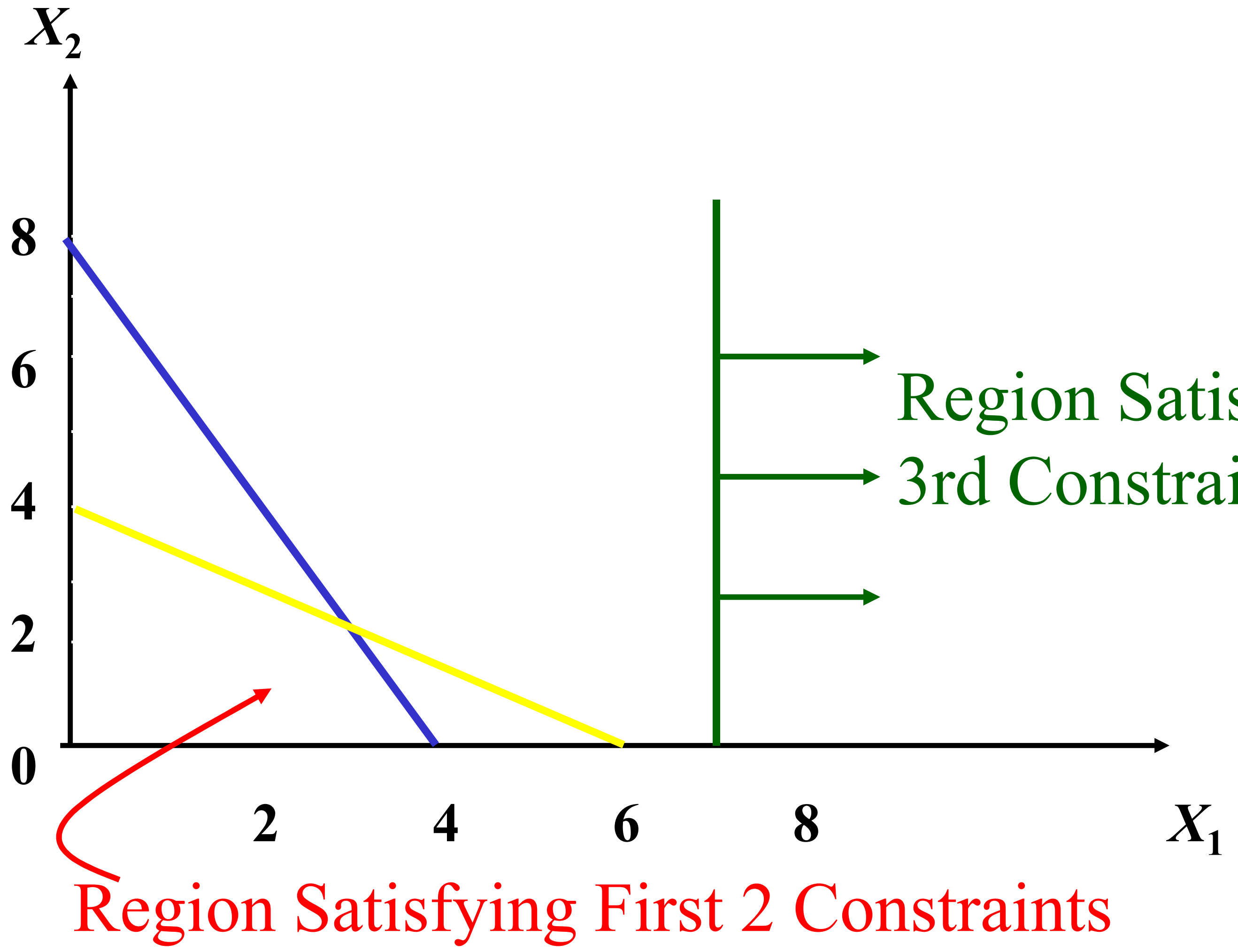


Special Cases in LP

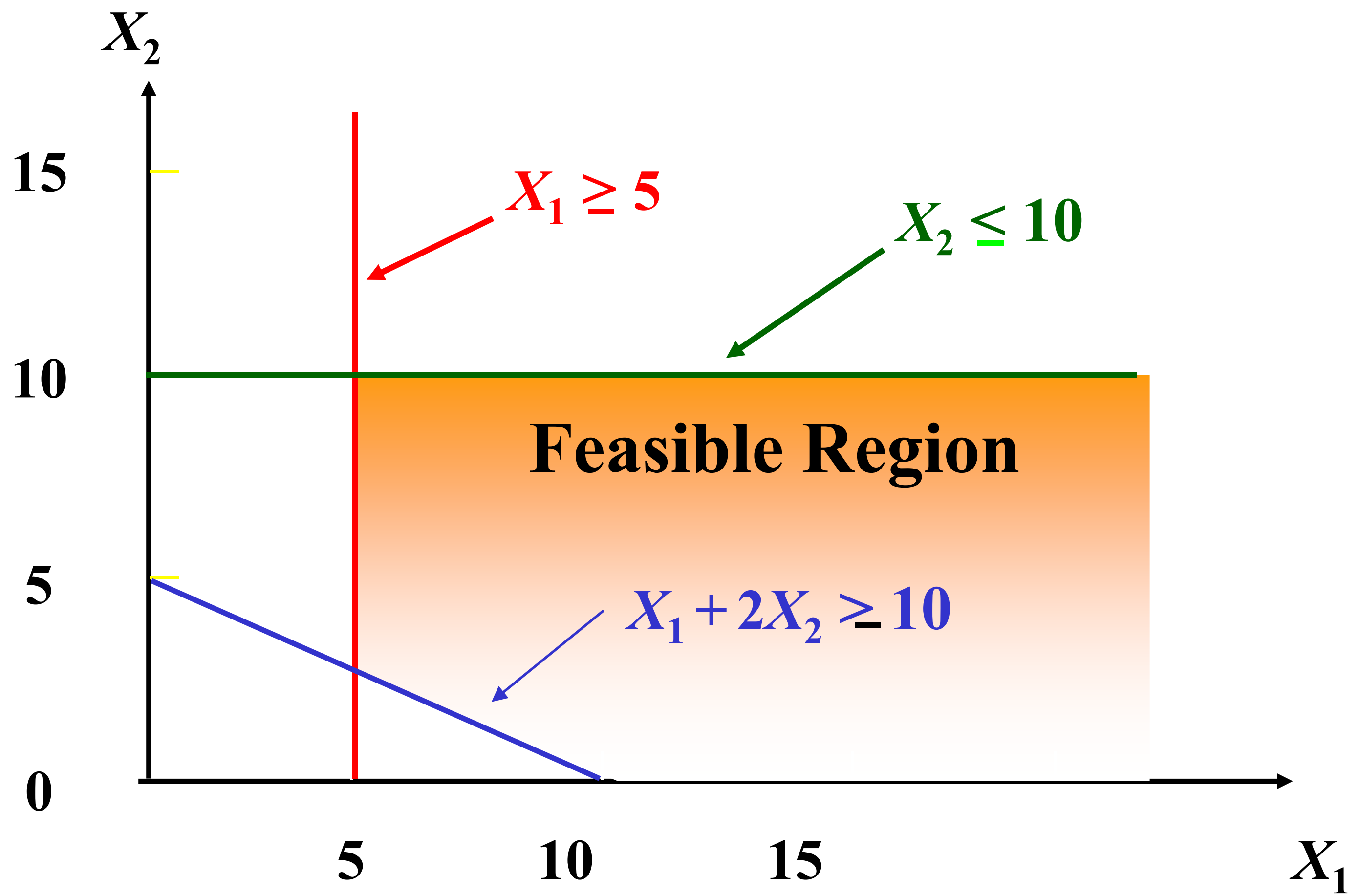
Four special cases and difficulties arise at times when solving LP problems:

1. Infeasibility:
   - *lack of a feasible solution region can occur if constraints conflict with one another.*
2. Unbounded Solutions:
   - *when the objective function in a maximization problem can be infinitely large, the problem is* unbounded *and is missing one or more constraints.*
3. Redundancy:
   - *a* redundant *constraint is one that does not affect the feasible solution region.*
4. More than One Optimal Solution:
   - *two or more optimal solutions may exist, and*
   - *this actually allows management great flexibility in deciding which combination to select.*

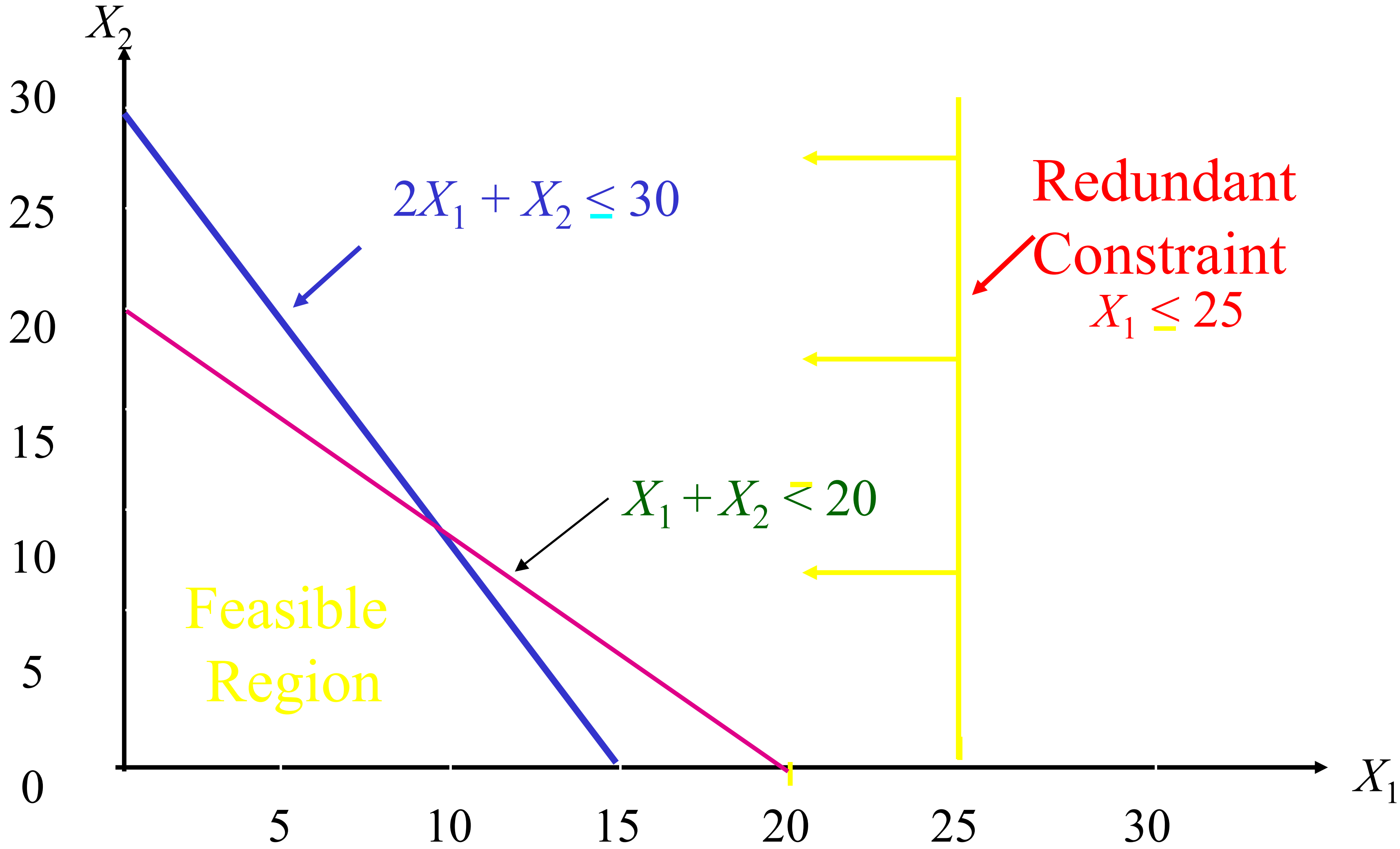
A Problem with No Feasible Solution



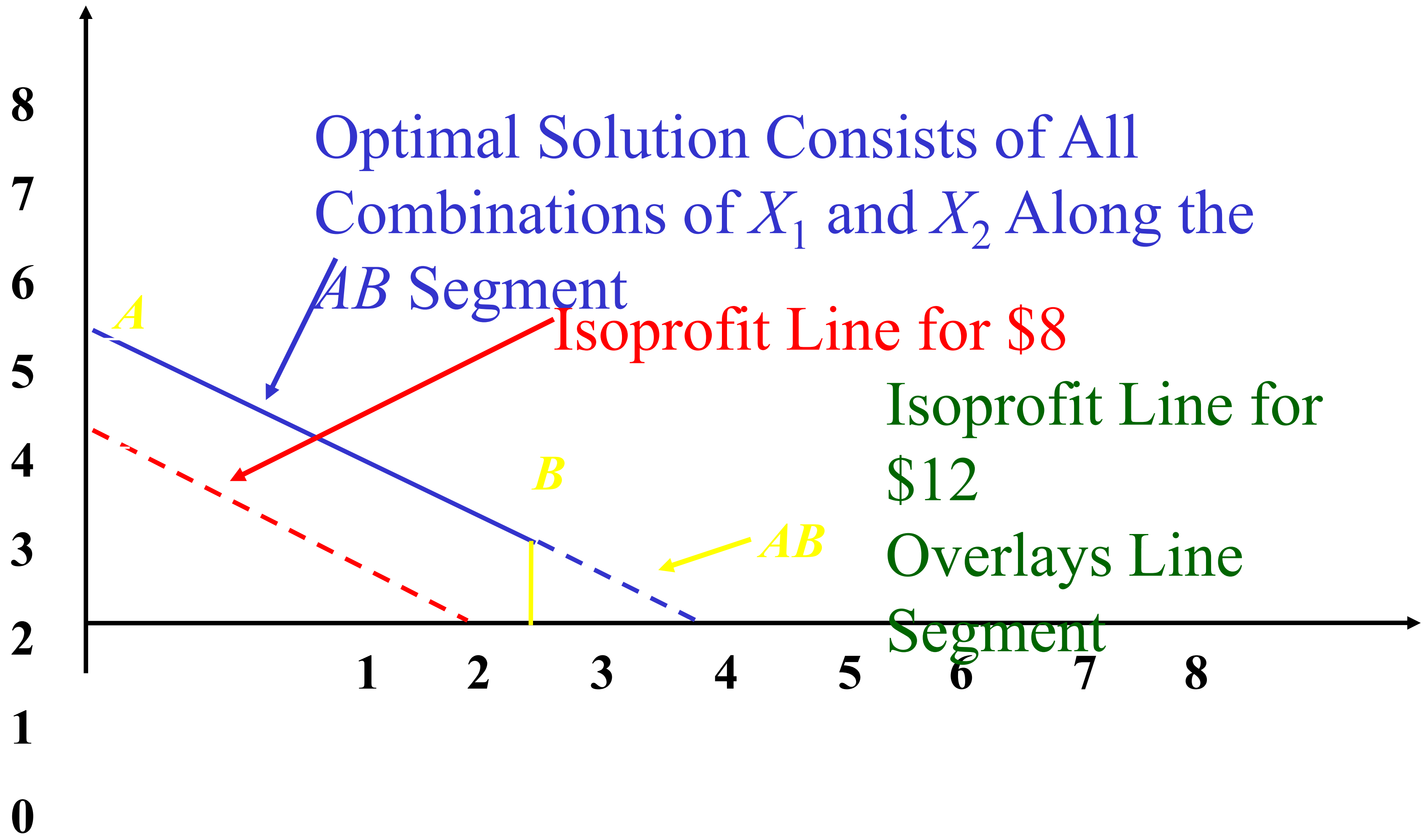
A Solution Region That is Unbounded to the Right



A Problem with a Redundant Constraint



An Example of Alternate Optimal Solutions



Sensitivity Analysis

- Optimal solutions to LP problems so far have been found under *deterministic assumptions*.
  - This means that we assume complete certainty in the data and relationships of a problem
  - i.e., prices are fixed, resources known, time needed to produce a unit exactly set.
- But in the real world, conditions are dynamic and changing.
- Questions to be addressed include:
  - How sensitive is the optimal solution to changes in profits, resources, or other input parameters?

Sensitivity Analysis

- One way to reconcile the discrepancy between the deterministic assumptions with dynamic and changing real-world conditions is to determine
  - how *sensitive* the optimal solution is to model assumptions and data.
- An important function of sensitivity analysis is to allow managers to experiment with values of the input parameters.
- Such analyses are used to examine the effects of changes in three areas:
  - contribution rates for each variable,
  - technological coefficients (the numbers in the constraint equations), and
  - available resources (the right-hand-side quantities in each constraint).

LINGO – MATHEMATICAL PROGRAMMIONG SOFTWARE

- **LINEAR PROGRAMMING**
- **INTEGER PROGRAMMING**
- **NON-LINEAR PROGRAMMING**

Marketing Applications

Media Selection

Medium	Audience Reached Per Ad	Cost Per Ad($)	Maximum Ads Per Week
TV spot (1 minute)	5,000	800	12
Daily newspaper (full page ad)	8,500	925	5
Radio spot (30 seconds, prime time)	2,400	290	25
Radio spot (1 minute, afternoon)	2,800	380	20